

Singular Perturbations in Quantum Field Theory*

V.E. Rochev and P.A. Saponov

Institute for High Energy Physics, Protvino, Russia

Abstract

In this talk we discuss a new approximation scheme for non-perturbative calculations in a quantum field theory which is based on the fact that the Schwinger equation of a quantum field model belongs to the class of singularly perturbed equations. The self-interacting scalar field and the Gross-Neveu model are taken as the examples and some non-perturbative solutions of an equation for the propagator are found for these models. The application to QCD is also discussed.

1 Preliminaries

It is well known that in solving an equation approximately one should distinguish two kinds of perturbations: singular and regular ones.

A perturbation is called to be singular if the term neglected at the leading approximation is (in some sense) the main term for a given equation. For example, in the case of differential equation such a term contains the highest derivative.

The perturbative solution for the singularly perturbed system can have nothing in common with the true solution of a given problem. As an elementary example, let us consider the following differential equation:

$$\lambda \dot{x} = t - x \tag{1}$$

supplied with the boundary condition: $x(0) = X$. The perturbative series for this problem consists of two terms

$$x_{pert}(t) \equiv \sum \lambda^n x_n = t - \lambda.$$

Though x_{pert} is an exact solution of differential equation (1) it does not give a solution of this problem if $X \neq -\lambda$. Indeed if we interested in values of derivatives of $x(t)$ at the origin ("vacuum expectation values") we can easily find from (1) that

$$\dot{x}(0) = -\frac{1}{\lambda}X, \quad \ddot{x}(0) = \frac{1}{\lambda^2}X + \frac{1}{\lambda}, \quad \text{etc.}$$

*The work is supported in part by the RFFI grant no. 95-02-03704a

That is the derivatives at the origin has definitely nonperturbative character when $X \neq -\lambda$ and therefore cannot be described by the perturbative series in λ . Such a situation is a consequence of an essential singularity in λ which enters the solution of this problem

$$x(t) = t - \lambda + (X + \lambda) \exp\left(-\frac{t}{\lambda}\right).$$

Many physical problems belong to the singularly perturbed class. One of the well known examples is the problem of the flowing of a viscous liquid near a boundary.

This process is described by the Navier-Stokes equation:

$$\nu \vec{\nabla}^2 \vec{v} = (\vec{v} \vec{\nabla}) \vec{v} + \frac{1}{\rho} \vec{\nabla} p$$

with the boundary condition fixed on the surface of the body being flowed around:

$$\vec{v}|_{\text{surf}} = 0.$$

Note that the viscosity ν is the coefficient at the highest derivative $\vec{\nabla}^2$. When $\nu = 0$ this equation goes into the Euler equation for the ideal liquid. At first sight it seems to be quite natural to take the Euler equation as a leading approximation to the case of small ν . However, solutions of the Euler equation cannot satisfy the boundary condition for the viscous liquid. The same is true for the perturbative theory, based on these solutions. The behaviour of the ideal liquid nearby the boundary is governed by weaker condition:

$$\vec{v}_\perp|_{\text{surf}} = 0.$$

That is why one should take into account the highest derivative term of Navier-Stokes equation from the very beginning, even if the viscosity ν is small.

Any quantum field theory model with interaction is also a typical example of singularly perturbed system. Let us turn to the simplest model of a scalar field $\phi(x)$, $x \in E_d$ with quartic selfinteraction:

$$\mathcal{L}_{int} = -\frac{\lambda}{4!} \phi^4.$$

In the quantum field theory one calculates the vacuum expectation values (n -point functions)

$$G_n(x_1 \dots x_n) = \langle 0 | T \phi(x_1) \dots \phi(x_n) | 0 \rangle.$$

These functions is the derivatives of the generating functional

$$G(j) = \sum \frac{1}{n!} G_n j^n$$

calculated at $j = 0$:

$$G_n = \left. \frac{\delta^n G}{\delta j^n} \right|_{j=0}.$$

The generating functional can be found as a solution of the Schwinger equation:

$$\frac{\lambda}{3!} \frac{\delta^3 G}{\delta j^3} + (m^2 - \partial^2) \frac{\delta G}{\delta j} - j G = 0.$$

so as λ is the coefficient at the highest derivative term, the perturbation theory in λ is singular. The same is true for any QFT model with interaction.

In the present report we discuss the new approximation scheme for the Schwinger equation which would take into account the singular character of this equation. The proposed scheme approximates the nonperturbative solutions of the Schwinger equation already at few first steps without summing expansions. This opens new possibilities for the description of essentially nonperturbative phenomena such as spontaneous symmetry breaking and others. For more detailed consideration of technical problems the readers are referred to [1] (see also [2]).

2 Approximation scheme

One of the key problems in finding a non-perturbative solution for the Schwinger equation is the problem of additional boundary conditions [2, 3, 4]. Indeed, the Schwinger equation being the differential equation of the order higher than one requires several boundary conditions in order to fix its solution uniquely. But generally one knows only one of them: the normalization of generating functional $G(0) = 1$. The other boundary conditions will determine the frames of physical phenomena which can be described by the corresponding solution of the Schwinger Equation. So as we do not know the explicit form of the additional boundary conditions we have to impose some constraints on the solutions of the Schwinger equation which would play the role of boundary conditions. In our approach such a constraint is the connected structures correspondence principle.

In the light of all mentioned above the perturbation theory acquires a peculiar role in the quantum field theory. Due to the Schwinger equation belongs to the class of singularly perturbed equations the iteration procedure of the perturbation theory can be closed without additional boundary condition. Fixing the generating functional norm is sufficient for the perturbation theory. As a result the perturbative solution fails to describe the physical phenomena which require the boundary conditions different from those automatically given by the iterative procedure of the perturbation theory. Nevertheless up to now the perturbation theory is the only universal tool for calculations in the quantum field theory.

So we will construct our scheme basing on the following requirements:

- i) The perturbative expansion in λ is always one of the possible solutions of the scheme;
- ii) Topological properties (connectivity) of the approximant to be found correspond to those of perturbative one.

The last requirement guarantees the true connected structure of the approximant and serves as an additional boundary condition which is necessary for obtaining a close system of the Dyson equations for the lowest Green functions.

Discussed below is the scheme for the theory $\lambda \phi_d^4$. The generalization to any other QFT model is rather obvious and does not contain principle difficulties.

Introduce the perturbative approximant

$$G_{pert}^N = \sum_{n=0}^N G^{(n)}, \quad G^{(n)} = O(\lambda^n),$$

which can be calculated in the standard way and *define* the approximant G^N as a solution of the equation:

$$\frac{\lambda}{3!} \frac{\delta^3 G^N}{\delta j^3} + (m^2 - \partial^2) \frac{\delta G^N}{\delta j} - j G^N = \frac{\lambda}{3!} \frac{\delta^3 G_{pert}^N}{\delta j^3} + (m^2 - \partial^2) \frac{\delta G_{pert}^N}{\delta j} - j G_{pert}^N. \quad (2)$$

When $N \rightarrow \infty$, $G_{pert}^N \rightarrow G_{pert} \equiv \sum_{n=0}^{\infty} G^{(n)}$. G_{pert} is an exact solution of the Schwinger equation therefore due to (2) G^N also tends to some exact solution of the Schwinger equation.

In accordance with the requirement ii) we will use the connected structures correspondence principle as the additional boundary condition for this equation. Namely, we will require the correspondence of the connected structures of G^N and G_{pert}^N .

To be more precise we should consider some set of approximants $\{G^N\}$ which obey equation (2) and connected structures correspondence principle. It is obvious that $G_{pert}^N \in \{G^N\}$ at any rate, therefore our scheme will be nontrivial if $\{G^N\}/G_{pert}^N \neq \emptyset$.

The choice of the proper solution should be made on the base of additional physical requirements such as the minimum of the ground state energy etc.

The connected structure of n -point function can be found from the theorem on the connectivity of the logarithm. A functional

$$Z(j) = \log G(j)$$

is a generating functional for the connected Green functions:

$$Z_n = \left. \frac{\delta^n Z}{\delta j^n} \right|_{j=0} = (G_n)^{con}.$$

For example

$$G_4 = 3 \triangle \triangle + G_4^{con},$$

where G_4 is the four-point function and \triangle is the *full* propagator:

$$\text{Diagram of } G_4 = \text{Diagram of } G_4^{con} + \text{other diagrams}$$

For the six-point function we have:

$$G_6 = 15 \triangle \triangle \triangle + 15 \triangle G_4^{con} + G_6^{con}$$

These relations are of the general type for the full Green functions.

The connected structure of the perturbative approximant possesses the following property at $O(\lambda^N)$ order of approximation:

$$\begin{aligned} (G_{2n}^N)_{pert}^{con} &\neq 0 \quad \text{if } n \leq N+1, \\ (G_{2n}^N)_{pert}^{con} &= 0 \quad \text{if } n > N+1. \end{aligned}$$

(In the model involved $G_{2n+1} = 0$ due to the parity conservation.) For example, at the order $O(1)$ (the free theory)

$$\begin{aligned} (G_2^0)_{pert}^{con} &= \triangle_c = (m^2 - \partial^2)^{-1}, \\ (G_{2n}^0)_{pert}^{con} &= 0 \quad n > 1. \end{aligned}$$

At the order $O(\lambda)$ we have

$$(G_2^1)_{pert}^{con} \neq 0, \quad (G_4^1)_{pert}^{con} \neq 0 \quad \text{and} \quad (G_{2n}^1)_{pert}^{con} = 0 \quad \text{if } n > 2, \text{ etc.}$$

This property will be the base of the connected structures correspondence principle. Namely let us require at the N -th step of approximation that

$$(G_{2(N+2)}^N)_{pert}^{con} = 0.$$

This equation allows us to obtain the relation:

$$G_{2(N+2)}^N = F \left[\triangle, G_4^N, \dots, G_{2(N+1)}^N \right]. \quad (3)$$

With the help of (3) we can close the system of $N+1$ Dyson equations for $N+1$ lowest functions $\triangle, G_4^N, \dots, G_{2(N+1)}^N$ at the N -th step of approximation. For example, at the first step ($N=0$)

$$(G_4^0)_{pert}^{con} = 0 \Rightarrow G_4^0 = 3 \triangle^0 \triangle^0$$

and we have one equation for the propagator Δ^0 . At the second step (N=1)

$$\left(G_6^1\right)^{con} = 0 \Rightarrow G_6^1 = 15 \Delta^1 \Delta^1 \Delta^1 + 15 \Delta^1 \left(G_4^1\right)^{con}$$

and we have the system of two equations for the propagator Δ^1 and the two-particle function G_4^1 and so on.

At any step of the scheme the system of these equations has at least one solution (the perturbative approximant). All other solutions (if they exist) are nonperturbative ones.

3 The first step: an equation for the propagator.

In this section we reproduce only the list of results for some models. Detailed derivation can be found in [1].

1. The theory ϕ_d^4 in Euclidean metric.

An equation for the propagator $\Delta^0 \equiv \Delta$ at the leading approximation has the form:

$$(m^2 - \partial^2) \Delta + \frac{\lambda}{2} \Delta \Delta = 1 + \frac{\lambda}{2} \Delta_c \Delta_c. \quad (4)$$

Besides the trivial perturbative solution $\Delta = \Delta_c$ there exist nonperturbative ones:

i) $d = 2$.

$$\Delta = \frac{1}{p^2 + \mu^2} + O(\lambda), \quad (5)$$

where $\mu^2 \simeq m^2 \exp\left(-\frac{8\pi m^2}{|\lambda|}\right)$, $\lambda \rightarrow -0$.

ii) $d = 4$.

There exist the solution of the form

$$\Delta = \frac{1}{\lambda} \frac{1}{p^2 + \mu^2} + O(1), \quad \mu^2 = O\left(\frac{1}{\lambda}\right), \quad \lambda \rightarrow +0, \quad (6)$$

and the dipole solution

$$\Delta = \frac{A}{p^2 + m^2} + \frac{B}{(p^2 + m^2)^2} \quad B = o_\lambda(A). \quad (7)$$

2. The Gross-Neveu model.

This is a model of a spinor field $\psi(x)$ in the two dimensional Minkowsky space with the Lagrangian:

$$\mathcal{L} = \bar{\psi} (i\hat{\partial} - m) \psi + \frac{\lambda}{2} (\bar{\psi}\psi)^2.$$

The equation for the propagator $S^0 = S$ reads as follows:

$$(m - i\hat{\partial}) S + i\lambda S S - i\lambda (Tr S) S = \mathbf{1} + i\lambda S_c S_c - i\lambda (Tr S_c) S_c. \quad (8)$$

Here $S_c = (m - i\hat{\partial})^{-1}$ is the free propagator.

Besides the trivial solution $S = S_c$ there exist nonperturbative solutions. At chiral limit $m \rightarrow 0$ they correspond to the spontaneous breakdown of the chiral symmetry. We would like to point out that in the original paper by Gross and Neveu the limit $N_c \rightarrow \infty$ for the N_c -component field was considered. In our approach the spontaneously broken solution exists for the one component field at small λ . This solution has the form

$$S(p) = \frac{4\pi}{\lambda} \frac{\hat{p} + \mu}{\mu^2 - p^2}, \quad \mu = O\left(\frac{1}{\lambda}\right). \quad (9)$$

3. QCD

The equations for the quark propagator and ghost propagator have only trivial perturbative solutions at the first step of approximation.

The equation for the gluon propagator takes the form:

$$(D^c)_{\mu\nu}^{-1} \mathcal{D}_{\mu\nu} + ig^2 C (\mathcal{D}_{\mu\lambda} \mathcal{D}_{\lambda\nu} - \mathcal{D}_{\lambda\lambda} \mathcal{D}_{\mu\nu}) = g_{\mu\nu} + ig^2 C (D_{\mu\lambda}^c D_{\lambda\nu}^c - D_{\lambda\lambda}^c D_{\mu\nu}^c), \quad (10)$$

where $D_{\mu\nu}^c$ is the free propagator and $\mathcal{D}_{\mu\nu}$ is the one to be sought for. The constant C is defined to be $\delta^{ab} C = f^{anc} f^{bnc}$. Besides the trivial solution $\mathcal{D} = D^c$ this equation has nonperturbative ones and among them the well-known "dual superconducting" solution of $1/p^4$ form. However, all these solutions are forbidden by the gauge invariance requirement. This is quite natural because at the first step only the four-gluon vertex contributes into the equation (10). One can expect, that nontrivial and physically interesting solutions (including the one with spontaneously broken chiral symmetry) will appear at the next step of approximation when three gluon, quark-gluon and ghost vertices will also play the game.

4 Conclusion

In conclusion we would like to emphasize once more that our scheme is not a variant of the perturbation theory partial summation. The perturbation theory (even being somehow summarized) cannot in principle exhaust the full information contained in the equations of the quantum field theory. The similar conclusions were made in the work [5] where the problem of nonunique solution to the Schwinger equation was approached from the different positions. So we hope that our scheme (or similar to it) can be useful in describing the nonperturbative content of quantum field models.

In this connection it is worth mentioning some ways for further development. First of all it would be interesting to find a nontrivial solution for the second step of the scheme which include a system of equations for the propagator and connected part of the full vertex (see [1]). Another crucial point of this approach is to develop criteria which could help to choose the proper solution from all the possible ones given by the scheme equations. And, of course, the main interest is to try to describe the nonperturbative phenomena in realistic physical theories such as QCD.

References

- [1] V.E. Rochev and P.A. Saponov, *Schwinger Equation as Singularly Perturbed Equation*, preprint IHEP 25-95, 1995; hep-th/9502142.
- [2] V.E. Rochev and P.A. Saponov, in Proc. IX Workshop "High Energy Physics and Quantum Field Theory" (Zvenigorod,1994), ed. B.B Levthchenko, Moscow Univ. Press, p. 313.
- [3] C.M. Bender, F. Cooper, and L.M. Simmons: *Phys. Rev. D*, **39**, 2343 (1989).
- [4] V.E. Rochev: *J.Phys.A.: Math. Gen.*, **26**, 1235 (1993).
- [5] S. Garsia and G.S. Guralnik, in Proc. Workshop "Quantum Infrared Physics" (Paris, 1994) eds. H.M. Fried and B. Müller, World Sci., Singapore, p. 205.